

Optimization and Decision Making

2018

Time allowed: 2 hours

*Full marks may be obtained for correct answers to
THREE of the FOUR questions.
All necessary working must be shown.*

1. Suppose that you have to choose an optimal portfolio with an overall expected rate of return of μ_{P+} from a list of n stocks and a risk-free asset. Stock i has expected rate of return μ_i with variance σ_i^2 for $i = 1, \dots, n$, and the covariance of the rates of return of stocks i and j is given by σ_{ij} for $i \neq j, i, j = 1, \dots, n$. The risk-free asset has rate of return ρ_0 . The proportion of stock i in the portfolio is denoted by w_i for $i = 1, \dots, n$.

- (a) What is the level of investment in the risk-free asset? What is the meaning of a negative value here? [3]
- (b) Formulate this as an optimization problem using Lagrange multipliers, and show that for the case when $n = 2$, the following linear equations hold for the optimal values of the w_i s,

$$\begin{aligned} w_1\sigma_1^2 + w_2\sigma_{12} - \lambda(\mu_1 - \rho_0) &= 0, \\ w_2\sigma_2^2 + w_1\sigma_{12} - \lambda(\mu_2 - \rho_0) &= 0, \\ -w_1\mu_1 - w_2\mu_2 - \rho_0(1 - w_1 - w_2) + \mu_{P+} &= 0. \end{aligned}$$

[10]

- (c) Now consider a problem with three stocks, including the risk free asset, where: $\mu_1 = 0.10, \mu_2 = 0.06, \sigma_1^2 = 0.2, \sigma_2^2 = 0.05, \sigma_{12} = 0.05, \rho_0 = 0.02$. Find the optimal portfolio for the expected rate of return of 0.08. [13]
- (d) Without solving the problem, discuss how your solution would change if the target expected rate of return changes. [4]

2. Three distribution centres send a given product to four supermarkets. The quantities that the distribution centres can deliver each week is limited because the manufacturers stock them weekly. The sales at each supermarket and the capacities of the distribution centres are as follows:

Centres A, B and C have stocks of 40, 60 and 100 respectively.

Supermarkets 1, 2, 3 and 4 require a supply of 30, 50, 50 and 60 respectively.

The transportation costs (in pounds) of supplying the outlets are given in the following table. The aim is to minimise the total cost of transporting the items.

	1	2	3	4
A	15	9	6	8
B	9	11	8	13
C	5	17	13	16

- (a) Formulate, but do not solve, the problem in the form of a linear program. [6]
- (b) Now write the problem in the form of a transportation array, explaining how you deal with the fact that the total demand and the total supply are not equal. [6]
- (c) Find the optimal transportation solution. [18]

3. A property company must decide whether to invest money in buying a plot of land and potentially developing it into a housing estate. It estimates that there is a 60% chance that planning permission will be obtained to develop the site within four years.

If planning permission has not been obtained at the end of this period, the company would abandon the project with a loss of £3 million.

If permission has been obtained, the company must decide the type of development, Luxury or Economy. The returns generated would depend upon the level of demand, categorized as either Strong or Weak.

If the company opted for Luxury development and demand was Strong, a £8 million profit would be obtained. If the demand was Weak in this case, however, the profit would be £1 million.

If the company decided to go for the Economy development, Strong demand would generate a profit of £5 million and Weak demand would generate a profit of £3 million.

The company estimates that there is a 45% chance that Strong demand will occur.

- (a) Construct a decision tree to represent this decision problem. [12]
- (b) Assuming that the company's objective is to maximize its expected profit, determine the policy that it should adopt. [4]
- (c) The company's estimated probability of Strong demand may be wrong. Assuming that all other problem elements remain the same, find the course of action for all possible probabilities of Strong demand. [4]
- (d) For the original problem in (b), Suppose that the probability of planning permission being granted is not fixed, but depends upon whether demand will be Strong or Weak, the planning authority being more likely to grant permission if they believe that there will be Strong demand. This leads to there being a 70% chance of permission being granted if there is Strong demand, but only a 40% chance for Weak demand. What is the optimal policy now? [6]
- (e) Before the final decision is taken the company is taken over by a new owner, who has utilities shown below for the sums of money involved in the decision. The new owner has no interest in other attributes which may be associated with the decision. What implications does this have for the policy that you found from (b) and why? [4]

Value (£ million)	-3	0	1	3	5	8
Utility	0	0.4	0.5	0.75	0.9	1

4. A jewelry manufacturer makes four different types of silver product; Rings, Necklaces, Bracelets and Figurines. Key to their brand is that they are small volume, and they only use (at most) a given volume of Silver each week. The manufacturer has broken down the requirements into two for each product, Labour of their craftspeople and Silver. The table below lists the four types, the Labour and Silver required for each product, the total current (weekly) supply of Labour and Silver to the manufacturer and the profit per item.

Product type	Ring	Necklace	Bracelet	Figurine	Supply
Silver (g)	24	40	30	60	2400
Labour (hours)	2	5	10	10	500
Profit (pounds)	100	240	300	400	

- (a) Set up the problem as a linear programme to find the maximum profit. [8]
- (b) Solve the problem using the simplex method. [12]
- (c) Suppose that the manufacturer also insists that a maximum of 65 items are produced each week. Set up and solve the new problem using the simplex method. [10]

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